

تمرین ۳

بخش ۱

فرض کنید $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ و $A \in M_{n,n}(\mathbb{C})$ باشند. کدام یک از عبارات زیر امکان پذیر هستند؟ با شرح مناسب برای هر گزینه مشخص کنید.

(a) $|\psi\rangle + \langle\phi|$

(h) $|\psi\rangle\langle\phi|A$

(b) $|\psi\rangle\langle\phi|$

(i) $|\psi\rangle A \langle\phi|$

(c) $A \langle\psi|$

(j) $\langle\psi|A|\phi\rangle$

(d) $|\psi\rangle A$

(k) $\langle\psi|A|\phi\rangle + \langle\psi|\phi\rangle$

(e) $\langle\psi|A$

(l) $\langle\psi|\phi\rangle\langle\psi|$

(f) $\langle\psi|A + \langle\phi|$

(m) $\langle\psi|\phi\rangle A$

(g) $|\psi\rangle|\phi\rangle$

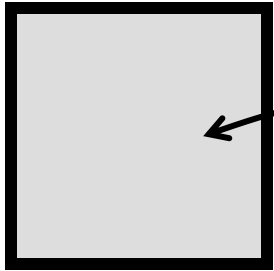
(n) $|\psi\rangle\langle\psi|\phi\rangle = \langle\psi|\phi\rangle|\psi\rangle$

بخش ۲

تمام «مثال‌هایی» که در ادامه در باکس‌های خاکستری رنگ آمده است را به صورت مختصر توضیح دهید.

مثلاً: مثال صفحه ۳ یک بردار احتمال است که احتمال حضور در سه حالت را نشان می‌دهد.

Rule 1



Machine has N states

$0, 1, 2, \dots, N-1$

N dimensional real vector

$$p \in \mathbb{R}^N$$

$$p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{N-1} \end{bmatrix}$$

positive elements

$$p_i \geq 0$$

which sum to unity

$$\sum_{i=0}^{N-1} p_i = 1$$

probability vector

Example: 3 state device

$$p = \begin{bmatrix} 0.3 \\ 0.7 \\ 0 \end{bmatrix}$$

30 % state 0

70 % state 1

0 % state 2

Rule 2

$$q_j = \sum_{i=0}^{N-1} A_{j,i} p_i$$

these are probabilities

$$A_{j,i} \geq 0$$
$$\sum_{j=0}^{N-1} A_{j,i} = 1$$

stochastic matrix

If in state 0 switch to state 0 with probability 0.4

If in state 0 switch to state 1 with probability 0.6

If in state 1 always stay in state 1

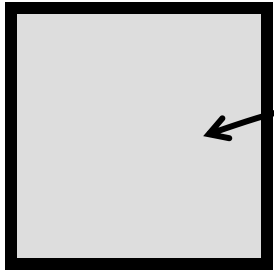
$$A = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 1 \end{bmatrix}$$

$$p = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

$$\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \rightarrow \begin{bmatrix} 0.16 \\ 0.82 \end{bmatrix}$$

$$q = Ap = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 1 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 0.84 \end{bmatrix}$$

Quantum Rule 1



Machine has N states

$0, 1, 2, \dots, N-1$

N dimensional complex vector (vector of amplitudes)

$$v \in \mathbb{C}^N \quad v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \quad v_i \in \mathbb{C}$$

Example: 2 state device

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

$$\sum_{i=0}^{N-1} |v_i|^2 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}} \frac{i}{\sqrt{2}} = 1$$

Quantum Rule 2, Example

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i & i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Conjugate:

$$U^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i & -i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Conjugate
transpose:

$$U^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix}$$

Unitary?

$$UU^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i & i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

evolves to

$$v' = Uv = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i & i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Computational Basis

Some special vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad |N-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example:

2 dimensional complex vectors (also known as: a qubit!)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computational Basis

Vectors can be “expanded” in the computational basis:

$$\begin{aligned} |v\rangle &= \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} = v_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + v_{N-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= v_0|0\rangle + v_1|1\rangle + \dots + v_{N-1}|N-1\rangle \end{aligned}$$

Example:

$$\begin{aligned} |v\rangle &= \begin{bmatrix} 1 + 2i \\ 3 \end{bmatrix} = (1 + 2i) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= (1 + 2i)|0\rangle + 3|1\rangle \end{aligned}$$

Computational Bras

Computational Basis, but now for bras:

$$\langle 0| = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\langle 1| = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}$$

\vdots

$$\langle N-1| = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\langle v| = \begin{bmatrix} v_0^* & v_1^* & \cdots & v_{N-1}^* \end{bmatrix} = v_0^* \langle 0| + v_1^* \langle 1| + \cdots + v_{N-1}^* \langle N-1|$$

Example:

$$\langle v| = \begin{bmatrix} 2 & 3 + 2i \end{bmatrix} = 2 \langle 0| + (3 + 2i) \langle 1|$$

The Inner Product

$$\langle w|v\rangle = w_0^*v_0 + w_1^*v_1 + \cdots + w_{N-1}^*v_{N-1}$$

Example:

$$|v\rangle = \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix}$$

$$|w\rangle = \begin{bmatrix} 3i \\ 3 \end{bmatrix}$$

$$\langle w| = \begin{bmatrix} -3i & 3 \end{bmatrix}$$

$$\langle w|v\rangle = \begin{bmatrix} -3i & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 + 2i \end{bmatrix} = (-3i) \cdot 1 + 3(1 + 2i) = 3 + 3i$$

$$\langle v|w\rangle = \begin{bmatrix} 1 & 1 - 2i \end{bmatrix} \begin{bmatrix} 3i \\ 3 \end{bmatrix} = 1 \cdot (3i) + (1 - 2i)3 = 3 - 3i$$

Complex conjugate of inner product: $(\langle w|v\rangle)^* = \langle v|w\rangle$

The Inner Product in Comp. Basis

$$\langle w|v\rangle = w_0^*v_0 + w_1^*v_1 + \cdots + w_{N-1}^*v_{N-1}$$

$$\langle w| = w_0^*\langle 0| + w_1^*\langle 1| + \cdots + w_{N-1}^*\langle N-1|$$

$$|v\rangle = v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle$$

$$\langle w|v\rangle = (w_0^*\langle 0| + w_1^*\langle 1| + \cdots + w_{N-1}^*\langle N-1|) (v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle)$$

$$\langle w|v\rangle = w_0^*v_0 + w_1^*v_1 + \cdots + w_{N-1}^*v_{N-1}$$

Handwritten notes:
 $|0\rangle\langle 0|$ $N \cdot 0^0$
 $w_0^*v_0 \langle 0|0\rangle +$
 $w_1^* \langle 0|1\rangle + \dots$

Example: $|v\rangle = |0\rangle + 2i|1\rangle$ $|w\rangle = 3i|0\rangle + (2i + 2)|1\rangle$

$$\langle w|v\rangle = -3i \cdot 1 + (-2i + 2)2i = 4 + i$$

Norm of a Vector

Norm of a vector:

$$||v\rangle|| = \sqrt{\langle v|v\rangle}$$

$$\begin{aligned}\langle v|v\rangle &= v_0^*v_0 + v_1^*v_1 + \cdots + v_{N-1}^*v_{N-1} \\ &= |v_0|^2 + |v_1|^2 + \cdots + |v_{N-1}|^2\end{aligned}$$

which is always a positive real number

it is the length of the complex vector

Example: $|v\rangle = |0\rangle + 2i|1\rangle$

$$\langle v|v\rangle = |1|^2 + |2i|^2 = 5$$

$$||v\rangle|| = \sqrt{5}$$

A Different Basis

A different orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\langle +|+\rangle = \left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = 1$$

$$\langle -|-\rangle = \left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{-1}{\sqrt{2}} \right|^2 = 1$$

$$\langle +|-\rangle = \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} \frac{-1}{\sqrt{2}} \right) = 0$$

An orthonormal basis is complete if the number of basis elements is equal to the dimension of the complex vector space.

Example Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = v_0|0\rangle + v_1|1\rangle$ in this basis: $|v\rangle = v_+|+\rangle + v_-|-\rangle$

$$\langle +|v\rangle = \langle +|(v_+|+\rangle + v_-|-\rangle) = v_+\langle +|+\rangle + v_-\langle +|-\rangle = v_+$$

$$\langle -|v\rangle = \langle -|(v_+|+\rangle + v_-|-\rangle) = v_+\langle -|+\rangle + v_-\langle -|-\rangle = v_-$$

So: $|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$

$$\langle +|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{1}{\sqrt{2}}v_1\right) = \frac{v_0 + v_1}{\sqrt{2}}$$

$$\langle -|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{-1}{\sqrt{2}}v_1\right) = \frac{v_0 - v_1}{\sqrt{2}}$$

$$|v\rangle = \frac{v_0 + v_1}{\sqrt{2}}|+\rangle + \frac{v_0 - v_1}{\sqrt{2}}|-\rangle$$

Explicit Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = |0\rangle$ in this basis: $|v\rangle = v_+|+\rangle + v_-|-\rangle$

$$\langle +|0\rangle = \frac{1}{\sqrt{2}}$$

$$|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$$

$$\langle -|0\rangle = \frac{1}{\sqrt{2}}$$

So:

$$|v\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix}$$

Matrices

A N dimensional complex matrix M is an N by N array of complex numbers:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$M_{j,k}$ are complex numbers

Example:

Three dimensional complex matrix:

$$M = \begin{bmatrix} 4 & 3 + i & 2 \\ i & e^{\frac{\pi}{4}} & \sqrt{2}i \\ 0 & 0 & 4 \end{bmatrix}$$

$$M_{1,0} = i$$

$$M_{2,2} = 4$$

Matrices, Multiplied by Scalar

Matrices can be multiplied by a complex number

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$\alpha M = \begin{bmatrix} \alpha M_{0,0} & \cdots & \alpha M_{0,N-1} \\ \vdots & & \vdots \\ \alpha M_{N-1,0} & \cdots & \alpha M_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3 + i \\ i & 1 \end{bmatrix}$ $\alpha = 2i$

$$\alpha M = \begin{bmatrix} 2i \cdot 0 & 2i(3 + i) \\ 2i(i) & 2i(1) \end{bmatrix} = \begin{bmatrix} 0 & -2 + 6i \\ -2 & 2i \end{bmatrix}$$

Matrices, Added

Matrices can be added

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \quad L = \begin{bmatrix} L_{0,0} & \cdots & L_{0,N-1} \\ \vdots & & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} \end{bmatrix}$$

$$M+L = \begin{bmatrix} M_{0,0} + L_{0,0} & \cdots & M_{0,N-1} + L_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} + L_{N-1,0} & \cdots & M_{N-1,N-1} + L_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3 + i \\ i & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 3 & -3 - i \\ i & 2 \end{bmatrix}$$

$$M+L = \begin{bmatrix} 0 + 3 & 3 + i + (-3 - i) \\ i + i & 1 + 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2i & 3 \end{bmatrix}$$

Matrices, Complex Conjugate

Given a matrix, we can form its complex conjugate by conjugating every element:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^* = \begin{bmatrix} M_{0,0}^* & \cdots & M_{0,N-1}^* \\ \vdots & & \vdots \\ M_{N-1,0}^* & \cdots & M_{N-1,N-1}^* \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3 + i \\ i & 1 \end{bmatrix}$$

$$M^* = \begin{bmatrix} 0 & 3 - i \\ -i & 1 \end{bmatrix}$$

Matrices, Transpose

Given a matrix, we can form its transpose by reflecting across the diagonal

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^T = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & & \vdots \\ M_{0,N-1} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3 + i \\ i & 1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 0 & i \\ 3 + i & 1 \end{bmatrix}$$

Matrices, Conjugate Transpose

Given a matrix, we can form its conjugate transpose by reflecting across the diagonal and conjugating

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^\dagger = \begin{bmatrix} M_{0,0}^* & \cdots & M_{N-1,0}^* \\ \vdots & & \vdots \\ M_{0,N-1}^* & \cdots & M_{N-1,N-1}^* \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3 + i \\ i & 1 \end{bmatrix}$$

$$M^\dagger = \begin{bmatrix} 0 & -i \\ 3 - i & 1 \end{bmatrix}$$