Quantum Information Processing

Lecture 12

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Fall 2022

January 10, 2023

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Public-key cryptography

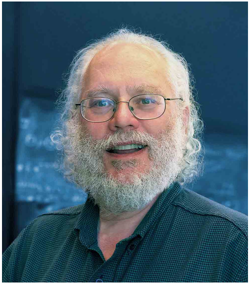
- In public-key cryptography Alice publishes a public key, which Bob uses to encode a message. Alice then uses her private key to decrypt the message.
- This relies on the asymmetry of the cryptography: it is easy to encrypt the message using the public key, but hard to decrypt it without knowledge of the private key.
- This in turn relies on the existence of one-way functions.

One-way functions

- One way function are functions that are easy to perform "forward", but hard to invert.
- For example, the RSA public-key cryptosystem uses factoring as a one-way function: multiplying two prime number p and q to give a composite number N is easy, but the inverse, factoring a composite number N into factors p and q is mathematically difficult.
- In fact, the best known classical algorithm for factoring, the number field sieve, requires $\exp(\Theta(n^{1/3} \log^{2/3} n))$ operations, where $n = \lceil \log_2 N \rceil$, i.e., the number of bits required to express N.
- For cryptographic applications it is crucial that the mathematical definition of one-way functions is that they must be hard to invert on average, not just in the worst case.

Factoring quantum mechanically

In 1994 Peter Shor invented a quantum algorithm which can factor numbers in polynomial time.
This remains one of the (or probably the) most important and impressive potential application of quantum computing.



Peter Shor

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Shor's factoring algorithm

For factoring a n-bit composite integer N.

1. Is N even? If so, output 2 and stop.
2. There is an efficient classical algorithm to check whether $N = c^l$ for some integers $c, l \geq 2$ and compute c if so. Run this classical algorithm and output c if obtained and stop.
3. If N is neither even nor a prime power, randomly choose $1 < x < N$, and compute $s = \gcd(x; N)$ (i.e., the greatest common divisor) using Euclids division algorithm. If $s \neq 1$ output s, which is a factor of N, and stop.
4. If $s = 1$ (i.e. x and N are co-prime), find the order r of the function $x \mod N$.
5. If r is odd, go back to step 3 and pick a different random number x. If r is even then perform efficient classical post-processing to extract a factor of N to output and stop.

Order finding

For co-prime positive integers x and N, such that $x < N$, the order of x modulo N is defined to be the least positive integer r such that $x^r = 1 \mod N$.

- Order finding is itself believed to be a hard problem classically.
- We will now show how quantum phase estimation can be used to find the order efficiently using a quantum computer.

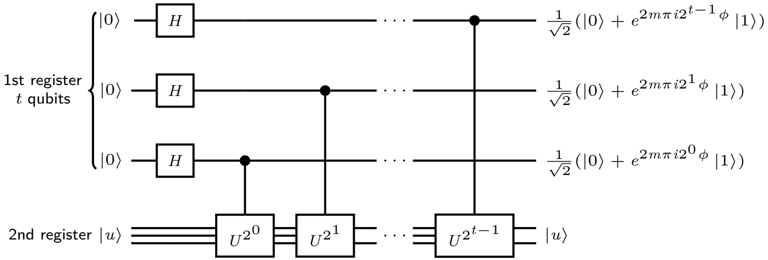
Order finding using quantum phase estimation

To find the order of x modulo N quantumly, we simply apply QPE using the unitary:

$$U|y\rangle = |(xy) \mod N\rangle$$

where y is an integer such that $0 \leq y < N$.
U is unitary when x and N are co-prime because U is merely a permutation matrix (i.e., it has exactly one 1 in each column and each row). To see this, consider the following argument (in which $y_1 \neq y_2$):
If $U|y_1\rangle = U|y_2\rangle$, then $xy_1 = xy_2 + kN$ for some integer $k \neq 0$. Therefore $y_1 - y_2 = \frac{kN}{x}$, i.e., $\frac{kN}{x}$ is an integer. However, as N and x are co-prime, the least (positive) integer k which satisfies this is $k = x$, i.e., $y_1 - y_2 = N$ (or $y_2 - y_1 = N$). So it cannot be the case that $0 \leq y_1, y_2 < N$.
So it follows, that for each integer y such that $0 \leq y < N$, $U|y\rangle$ gives a different integer between 0 and $N - 1$ inclusive, so U is a permutation matrix.
For completeness, for $N \leq y < 2n$, we define $U|y\rangle \to y$.

Recall that the QPE circuit is thus:



To perform QPE we need to implement controlled-U^{2^j} gates and prepare the second register in an eigenstate of U. However, regarding the first of these, if we were simply to take a naive approach, we would incur an exponential amount of operations.

Implementing the series of controlled-U^{2^j} gates

There are a few variations on the method used to implement the series of controlled-U^{2^j} gates, so here we present one which can be (relatively) easily understood. We begin by noting:
Therefore:

1. We first pre-compute x^{2^j} mod N for all $j, 0 \leq j \leq t-1$ (this is known as modular exponentiation).
2. We then use these pre-computed values to implement the unitary operation $|y\rangle \rightarrow |((x^{2^j} \mod N)y) \mod N\rangle$ (controlled as appropriate) for all $j, 0 \leq j \leq t-1$ in sequence as required.

Complexity of implementing the controlled-U^{2^j} gate

Noting that multiplication takes $O(m^2)$ operations, where m is the number of bits needed to specify the numbers being multiplied, the computational complexity of the two steps required to implement the controlled-U^{2^j} gates is therefore as follows:

1. Pre-computing $x^{2^j} \mod N$ for all $j, 0 \leq j \leq t-1$ can be achieved by repeated modular squaring of x - i.e., a total of t squaring operations. Note that $x^{2^j} \mod N < N$, so a maximum of n bits are needed to specify the numbers being squared, so the total number of operations required is $O(n^2 t)$.
2. With the pre-computed values of $x^{2^j} \mod N$ we need to implement the multiplication $((x^{2^j} \mod N)y) \mod N$ for all j, $0 \leq j \leq t-1$, i.e., t times in total. $y < N$, so a total of n bits are required to express each of the numbers being multiplied, therefore the complexity of this step is also $O(n^2 t)$.

To extract the order from the phase it suffices that $t \in O(n)$, so putting these together we have that the number of operations required to perform the series of controlled-U^{2^j} gates is $O(n^3)$.

The eigenvalues of U

QPE also requires that we prepare the input to the series of controlled-U^{2^j} gates in an eigenstate of U. States defined by:

$$|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} |x^k \mod N\rangle$$

for $0 \leq s \leq r-1$ are eigenstates of U, because:

$$\begin{aligned}
U|u_s\rangle &= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} |x^{k+1} \mod N\rangle \\
&= \frac{1}{\sqrt{r}} \sum_{k=1}^{r} e^{-2\pi i s (k-1)/r} |x^{k} \mod N\rangle \\
&= e^{2\pi i s/r} \frac{1}{\sqrt{r}} \sum_{k=1}^{r} e^{-2\pi i s k/r} |x^{k} \mod N\rangle \\
&= e^{2\pi i s/r} \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} |x^{k} \mod N\rangle \\
&= e^{2\pi i s/r} |u_s\rangle
\end{aligned}$$

A superposition of eigenstates of U

To prepare an eigenstate as defined on the previous slide requires knowledge of r, which we clearly dont have, so instead we prepare an equal superposition of eigenstates for $0 \leq s \leq r-1$:

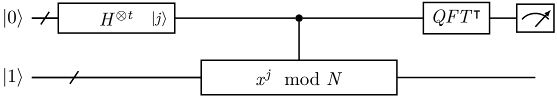
$$\begin{aligned}\frac{1}{\sqrt{r}}\sum_{k=0}^{r-1}|u_s\rangle &= \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}\sum_{k=0}^{r-1}e^{-2\pi isk/r}|x^k \mod N\rangle \\ &= \frac{1}{r}\left(\sum_{s=0}^{r-1}e^0|x^0 \mod N\rangle + \sum_{k=1}^{r-1}\left(\sum_{s=0}^{r-1}e^{-2\pi isk/r}\right)|x^k \mod N\rangle\right) \\ &= \frac{1}{r}r|1\rangle \\ &= |1\rangle\end{aligned}$$

Which is a state that can be prepared easily.

Order finding

Quantum circuit

We previously saw that $|u_s\rangle$ has eigenvalue $e^{2\pi isk/r}$ - i.e., its phase is s/r, so we now run QPE, with the second register in the state $|1\rangle$ (note this is $|1\rangle$ such that 1 is an n-bit binary number, rather than simply $[0,1]^{\mathsf{T}}$).



The measurements will collapse the state into one of the eigenstate components of $|1\rangle$ because each digital eigenvalue phase estimation in the first register will be entangled with its corresponding eigenvector in the second register:

$$\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|0\rangle^{\otimes n}|u_s\rangle \rightarrow \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}\widetilde{s/r}|u_s\rangle$$

Thus QPE returns an estimate of the phase $\widetilde{s/r}$ for some (unknown) integer s, however there is a classical algorithm (the continued fractions algorithm) that can extract r from $\widetilde{s/r}$ with high probability.

Reduction of factoring to order-finding

From the order-finding subroutine we have found r such that $x^r \mod N = 1$, i.e.,

$$(x^r - 1) \mod N = 0$$

If r is even this factorises:

$$(x^{r/2} - 1)(x^{r/2} + 1) \mod N = 0$$

For which there are four possibilities:

1. $x^{r/2} = 1 \mod N$: but we know this cannot actually occur, as r is the least integer satisfying $x^r = 1 \mod N$.
2. $x^{r/2} = -1 \mod N$: in which case the algorithm fails.
3. $(x^{r/2} - 1)(x^{r/2} + 1) = N$ in which case $(x^{r/2} - 1)$ and $(x^{r/2} + 1)$ are factors that we output.
4. $(x^{r/2} - 1)(x^{r/2} + 1) = kN$ for some $k \geq 2$, in which case there is a result that guarantees that one of $\gcd((x^{r/2} + 1); N)$ or $\gcd((x^{r/2} - 1); N)$ is a non-trivial factor of N, which we can run Euclids algorithm to find and then output.

A single run of Shor's algorithm only returns a factor with a certain probability. In particular:

1. The order-finding subroutine could return even r.
2. The order-finding subroutine could return r such that $x^{r/2} = -1 \mod N$.
3. It is possible that the classical post-processing required to extract the order from the phase can fail.

The probability that r is such that neither of the first two occur is at least one half. The probability that the order-finding subroutine is such that the classical post-processing correctly returns the order is at least $1 - \epsilon$ for some constant ϵ.
Therefore a single run of Shor's algorithm correctly returns a factor with probability $O(1)$, which is acceptable, as the expected number of iterations needed to find a factor does not grow with n.

Shor's algorithm

Computational complexity

Recall that the best classical algorithm for factoring requires $\exp(\Theta(n^{1/3}\log^{2/3} n))$ operations.

- Shor's algorithm (as we have expressed it here) calls two classical algorithms as subroutines - the prime power checker, and Euclid's algorithm - both of which require a number of operations that is only polynomial in n.
- The quantum circuit used in Shor's algorithm requires $O(n^3)$ gates to perform the modular exponentiation, and $O(n^2)$ gates to perform the QFT.
- As the success probability of a single run of Shor's algorithm is $O(1)$, the number of iterations we expect does not grow with n.

Therefore Shor's algorithm can factor in a number of operations that is polynomial in the number of bits required to express the number being factored. An exponential speed-up compared to the best classical algorithm.