Quantum Information Processing

Lecture 7

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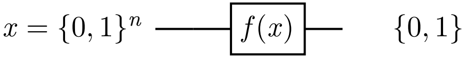
In order to design quantum algorithms we need to know how to execute functions on a quantum computer. In particular, we are interested in functions that take a (binary) number, and output a truth value (i.e., $\{0,1\}$). That is, functions of the form $f : \{0,1\}^n \to \{0,1\}$.



However, we know that a quantum circuit is composed of quantum gates, which are unitary operations, and apart from for trivial functions, this is not a unitary.

Binary numbers and quantum states

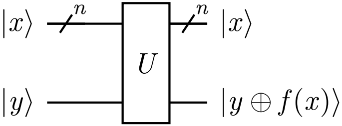
It is usually convenient to express the action of a quantum circuit for a general quantum state and then use linearity to express a sum of superposed terms if necessary. To do so, we analyse the circuit for a general n-qubit state:

$$|x\rangle = |x_1 x_2 \dots x_n\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

where $x_i \in \{0, 1\}$ for $i = 1 \dots n$ and associated with $|x\rangle$ is the n-bit binary number x.

Arbitrary mathematical functions as unitaries

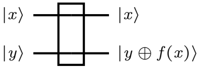
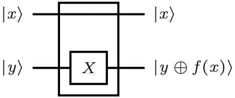
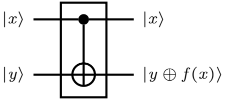
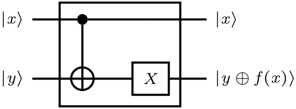
Instead, we must use the same trick that allowed us to write down the Toffoli gates as a quantum (unitary) version of the classical AND gate: as well as evaluating the function we output the input data:



U can easily be seen to be self-inverse, and the universality of quantum computing implies that any function $f(x)$ can be efficiently encoded in this way (to desired accuracy) as a quantum circuit consisting of gates from a finite universal set.

Constant and balanced function on a single bit

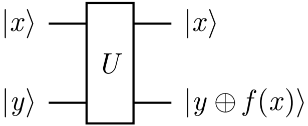
If the input, x, is a single bit (as is the output), then we have four possible functions:

Function	x	f(x)	Type	Unitary
$f(x) = 0$	0 1	0 0	Constant	
$f(x) = 1$	0 1	1 1	Constant	
$f(x) = x$	0 1	0 1	Balanced	
$f(x) = x \oplus 1$	0 1	1 0	Balanced	

Deutsch's algorithm set-up

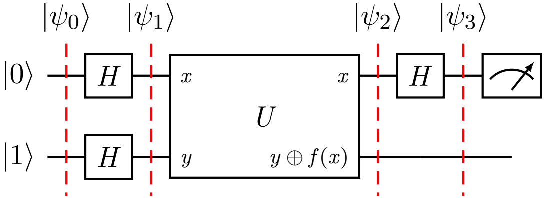
We want to find out whether a particular function, with one input bit and one output bit is constant or balanced. Classically, we need to evaluate the function twice (i.e., for input $= 0$ and input $= 1$), but remarkably, we only need to evaluate the function once quantumly, by using Deutsch's algorithm.

We have a two qubit unitary, which is one of the four on the previous slide (we don't know which):



Which we are going to incorporate into a quantum circuit.

Deutsch's algorithm (1)



Initially we prepare the state:

$$|\psi_0\rangle = |01\rangle$$

Which the initial Hadamard gates put in the superposition state:

$$\begin{aligned}|\psi_1\rangle &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\ &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)\end{aligned}$$

Deutsch's algorithm (2)

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

Next the unitary is implemented, which sets the second qubit to $y \oplus f(x)$, so we have four options for $|\psi_2\rangle$:

$$f(x) = 0 \qquad |\psi_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$f(x) = 1 \qquad |\psi_2\rangle = \frac{1}{2}(|01\rangle - |00\rangle + |11\rangle - |10\rangle)$$

$$f(x) = x \qquad |\psi_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle)$$

$$f(x) = x \oplus 1 \qquad |\psi_2\rangle = \frac{1}{2}(|01\rangle - |00\rangle + |10\rangle - |11\rangle)$$

which factorises as:

$$|\psi_2\rangle = \begin{cases} \pm\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0) = f(1) \\ \pm\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0) \neq f(1) \end{cases}$$

That is the two balanced cases differ only by an unobservable global phase (and likewise for the two constant cases).

$$|\psi_2\rangle = \begin{cases} \pm\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0)=f(1) \\ \pm\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0)\neq f(1) \end{cases}$$

The next step is to use the Hadamard gate to interfere the superposition on the first qubit, which yields:

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0)=f(1) \\ \pm|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right), & \text{if } f(0)\neq f(1) \end{cases}$$

The final step is to measure the first qubit, and we can see that the outcome will always be 0 if the function is constant, and 1 if balanced. We can see that superposition and interference, in some sense, play complementary roles: we prepare a state in superposition, perform some operations, and then use interference to discern some global property of the state.

Quantum computing jargon

Query complexity:

- In Deutsch's algorithm we are not using a quantum computer to evaluate a "classically difficult" function per se, but rather using quantum phenomena to reduce the number of queries we need to make to an unknown function, to ascertain some information thereabout.

Oracles and black boxes

- In Deutsch's algorithm, and other query complexity algorithms, we query U, which is known as a "black box", or often in quantum computing an "oracle". The oracle in Deutsch's algorithm is sufficiently simple that we can explicitly express each possible option, but frequently in quantum computing problems are framed in terms of oracles, even when this is not the case.

Deutsch-Jozsa algorithm

Richard Jozsa

Together with Richard Jozsa, who is now a Professor at DAMTP, David Deutsch generalised the algorithm to apply to constant/balanced functions of any input size.

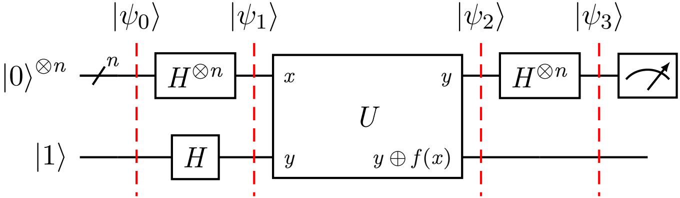
Deutsch's problem

The Deutsch-Jozsa algorithm is usually motivated in terms of Deutsch's problem:

- We have a function $f : \{0,1\}^n \rightarrow \{0,1\}$.
- We are promised this function is either constant (same output for each x) or balanced ($f(x)$ is equal to each of 0 and 1 for exactly half of the possible values of x).
- Classically, we can see that we may need to query the function $\frac{2^n}{2} + 1$ times to be sure whether the function is constant or balanced.
- But quantumly, if the function is encoded as a quantum oracle, then the Deutsch-Jozsa algorithm allows us to determine whether the function is constant or balanced with only a single oracle call.

Deutsch-Jozsa algorithm (1)

The circuit of the Deutsch-Jozsa algorithm closely resembles that of Deutsch's algorithm:



The initial state of which can be expressed:

$$|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$$

which is then put into superposition, which can conveniently be expressed:

$$|\psi_1\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^{n+1}}}|x\rangle(|0\rangle - |1\rangle)$$

Deutsch-Jozsa algorithm (2)

$$|\psi_1\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^{n+1}}}|x\rangle(|0\rangle - |1\rangle)$$

The unitary transforms $|\psi_1\rangle$ to:

$$|\psi_2\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^{n+1}}}(-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)$$

We now address the interference $H^{\otimes n}$ on the first n wires, for which we use the expression:

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x.z}|z\rangle$$

which allows us to express:

$$|\psi_3\rangle = \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} \frac{1}{2^n}(-1)^{x.z+f(x)}|z\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Deutsch-Jozsa algorithm (3)

We can now determine whether the function is constant or balanced by measuring the first n qubits of the final state (i.e., we neglect the final qubit which is in the $|-\rangle$ state):

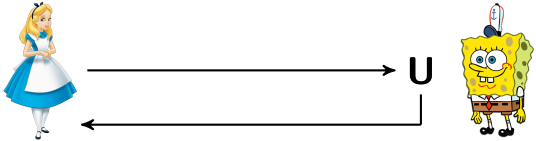
$$|\psi_3\rangle = \left(\sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{1}{2^n} (-1)^{x.z+f(x)} |z\rangle \right) \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Specifically, we consider the probability of measuring zero on every qubit, which corresponds to the term in the superposition where $|z\rangle$ is $|0\rangle^{\otimes n}$:

- In the case where the function is constant, then the coefficient of $|0\rangle^{\otimes n}$, $\sum_x (-1)^{f(x)}/2^n$ is equal to ± 1 ... as this has amplitude 1, then we measure $|0\rangle^{\otimes n}$ with probability one.
- In the case where the function is balanced then $\sum_x (-1)^{f(x)}/2^n = 0$, and so we will never measure $|0\rangle^{\otimes n}$.

So it follows that measuring the first n qubits allows us to determine with certainty whether the function is constant (measure all zeros) or balanced (measure at least one 1).

Potential for exponential speed-up using a quantum computer



Imagine now that the oracle is held by a person, “Bob”, who is spatially separated from the person, “Alice”, who is trying to determine whether the function is constant or balanced.

1. To resolve an instance of Deutschs problem, classically Alice transmits $\frac{2^n}{2}+1$ messages, each of size n bits, and each of which Bob replies to with a one bit message.
2. Whereas quantumly the Deutsch-Jozsa algorithm requires only the transmission of a single $n+1$ qubit message by Alice, to which Bob replies with a n qubit message.

So there is an exponential reduction in the amount of information transfer required to solve an instance of Deutsch’s problem.